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Graphs with Large Roman Domination Number

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ABSTRACT

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex with label 0 is adjacent to a vertex with label 2. The weight of an RDF f is $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of G is the minimum weight of an RDF in G. In this article, we characterize all connected graphs G of order n whose Roman domination number is $n-1$ or $n-2$.

Keywords: Roman dominating function, Roman domination number.

1. Introduction

Throughout this article, we only consider finite connected graph G with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). A graph is simple if it has no loops and no two of its links join the same pair of vertices. For every vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. The *minimum* and *maximum* degree of G are respectively denoted by $\delta(G)$ and $\Delta(G)$.

A *tree* is an acyclic connected graph, a *leaf* of a tree T is a vertex of degree 1, and a *stem* is a vertex adjacent to a leaf. For two integers $r, s \geq 1$, a double star $S(r, s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves.

We write K_n for the *complete graph* of order n, C_n for a *cycle* of length n and P_n for a *path* of length $n-1$. We apply the notation $cor(F)$ for *corona* of a graph F, which obtained from F by attaching a leaf to each vertex of F. The induced subgraph by $S \subseteq V(G)$ is denoted by $G[S]$. The *cartesian product* $G = H \Box K$ has $V(G) = V(H) \times V(K)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(K)$ or $u_2 = v_2$ and $u_1v_1 \in E(H)$. We use West (2001) for terminology and notation which are not defined here.

A *Roman dominating function* (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex with label 0 is adjacent to a vertex with label 2. The weight of an RDF f is $w(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of G, $\gamma_R(G)$, is the minimum weight of an RDF in G; that is, $\gamma_R(G) = min\{w(f) | f \text{ is an RDF in } G\}$. An RDF of weight $\gamma_R(G)$ is called a $\gamma_R(G)$-function. The concept of Roman domination in graphs was introduced by Cockayne et al. (2004) and further studied in for example in Ahangar et al. (2014), Chambers et al. (2009), Cockayne et al. (2005), Fernau (2008), Haynes et al. (1998a,b), Henning (2002), Song and Wang (2006).

The aim of this article is to characterize all connected graphs with large Roman domination number. All graphs G of order n with $\gamma_R(G) = n$ are characterized in Cockayne et al. (2004). In the next section, we continue the study of Roman dominating function and we characterize all graphs G of order n for which $\gamma_R(G) = n - 1$ or $\gamma_R(G) = n - 2$.

We make use of the following results.

Proposition 1.1 (Chambers et al. (2009))**.** *For a connected graph G of order n, $\gamma_R(G) \leq n - \Delta(G) + 1$.*

Proposition 1.2 (Chambers et al. (2009))**.** *For a connected graph G of order $n \geq 3$, $\gamma_R(G) \leq 4n/5$.*

Proposition 1.3 (Cockayne et al. (2004))**.** *For paths and cycles, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proposition 1.4 (Chambers et al. (2009))**.** If G is a connected n-vertex graph with $\delta(G) \geq 2$ other than C_4, C_5, C_8, F_1, F_2, then $\gamma_R(G) \leq \frac{8n}{11}$, where graphs F_1 and F_2 are given in Figure 1.

Proposition 1.5 (Cockayne et al. (2004))**.** If G is a connected n-vertex graph, then $\gamma_R(G) \leq n\frac{2+\ln(\frac{1+\delta(G)}{2})}{1+\delta(G)}$.

2. Main result

In this section we characterize all connected graphs of order n for which $\gamma_R(G) = n - 1$ or $\gamma_R(G) = n - 2$.

Proposition 2.1. *Let G be a graph of order n. Then $\gamma_R(G) = n - 1$ if and only if $G \in \{C_3, C_4, C_5, P_3, P_4, P_5\}$.*

Proof. Suppose $\gamma_R(G) = n - 1$. From Proposition 1.1, we obtain $\Delta(G) \leq 2$, and from Proposition 1.2 we obtain that $n \leq 5$. If $\Delta(G) = 1$ then $G = K_2$, a contradiction. Thus $\Delta(G) = 2$, and so G is either a path or a cycle. Now the result follows by Proposition 1.2. The converse is obvious. □

Here we characterize the graphs G with the properties that $\gamma_R(G) = 2$, $\gamma_R(G) = 3$, $\gamma_R(G) = 4$ or $\gamma_R(G) = 5$.

Proposition 2.2. (i) For a graph G of order $n \geq 2$, $\gamma_R(G) = 2$ if and only if $\Delta(G) = n - 1$ or $n = 2$.

(ii) For a graph G of order $n \geq 3$, $\gamma_R(G) = 3$ if and only if (a) $n = 3$ and $\Delta(G) \leq 1$ or (b) $\Delta(G) = n - 2$.

(iii) For a graph G of order $n \geq 4$, $\gamma_R(G) = 4$ if and only if (a) $\Delta(G) = n - 3$ or (b) $\Delta(G) \leq n - 4$ and there are two vertices $u, v \in V(G)$ such that $N[u] \cup N[v] = V(G)$ or (c) $n = 4$ and $\Delta(G) \leq 1$.

(iv) For a graph G of order $n \geq 5$, $\gamma_R(G) = 5$ if and only if $\Delta(G) \leq n-4$ and $N[u] \cup N[v] \subsetneqq V(G)$ for all pairs of vertices $u, v \in V(G)$. In addition, (a) there are two vertices $u, v \in V(G)$ such that $|N[u] \cup N[v]| = n-1$ or (b) $n = 5$ and $\Delta(G) \leq 1$ or (c) G contains a vertex w with $\deg(w) = n-4$ and $\Delta(G[V(G) - N[w]]) \leq 1$.

Proof. Since the proof of (i) is trivial, we omit it.

(ii) Assume G has no vertex v with $\deg(v) = n-1$. It follows from (i) that $\gamma_R(G) \geq 3$. The other two assumptions show that $\gamma_R(G) \leq 3$, and so we obtain $\gamma_R(G) = 3$.

Conversely, suppose that $\gamma_R(G) = 3$. It follows from (i) that G has no vertex v with $\deg(v) = n-1$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $V_2 = \emptyset$, then $|V_1| = 3 = n$ and thus (a) holds. If $V_2 \neq \emptyset$, then $|V_1| = |V_2| = 1$. Suppose $V_2 = \{v\}$. Then $uv \in E(G)$ for each $u \in V_0$ and hence $\deg(v) = n-2$. Hence, condition (b) is proved.

(iii) If G satisfies (a), (b) or (c), then clearly $\gamma_R(G) \leq 4$. We deduce from (i) and (ii) that $\gamma_R(G) = 4$.

Conversely, let $\gamma_R(G) = 4$. It follows from (i) and (ii) that $\Delta(G) \leq n-3$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $V_2 = \emptyset$, then $n = |V_1| = 4$ and so (c) holds. Assume $n \geq 5$. We distinguish two cases.

Case iii-1: Assume that $|V_2| = 1$ and $|V_1| = 2$. If $V_2 = \{v\}$, then we conclude that $\deg(v) = n-3$ and the condition (a) holds.

Case iii-2: Assume that $|V_2| = 2$. If $V_2 = \{u, v\}$, then we have $N[u] \cup N[v] = V(G)$, and we obtain condition (b).

(iv) By (i), (ii), (iii), the conditions $\Delta(G) \leq n-4$ and $N[u] \cup N[v] \subsetneqq V(G)$ for all pairs of vertices $u, v \in V(G)$, imply that $\gamma_R(G) \geq 5$. The other three assumptions show that $\gamma_R(G) \leq 5$, and hence we obtain $\gamma_R(G) = 5$.

Conversely, assume that $\gamma_R(G) = 5$. Using (i), (ii) and (iii), we can see that $\Delta(G) \leq n-4$ and $N[u] \cup N[v] \subsetneqq V(G)$ for all pairs of vertices $u, v \in V(G)$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(G)$-function. If $V_2 = \emptyset$, then $|V_1| = 5$ and thus $n = 5$ and $\Delta(G) \leq 1$. Hence, G satisfies (b). Let $V_2 \neq \emptyset$. Again, we distinguish two cases.

Case iv-1: Assume that $|V_2| = 1$ and $|V_1| = 3$. If $V_2 = \{w\}$, then we deduce that $\deg(w) = n - 4$. Since $N[u] \cup N[v] \subsetneq V(G)$ for all pairs of vertices $u, v \in V(G)$, we deduce that $\Delta(G[V(G) - N[w]]) \leq 1$ and so G satisfies (c).

Case iv-2: Assume that $|V_2| = 2$ and $|V_1| = 1$. If $V_2 = \{u, v\}$, then it follows that $|N[u] \cup N[v]| = n - 1$ and condition (a) is proved. $\square$

Let $\mathcal{F}$ be the family of graphs illustrated in Figure 1.

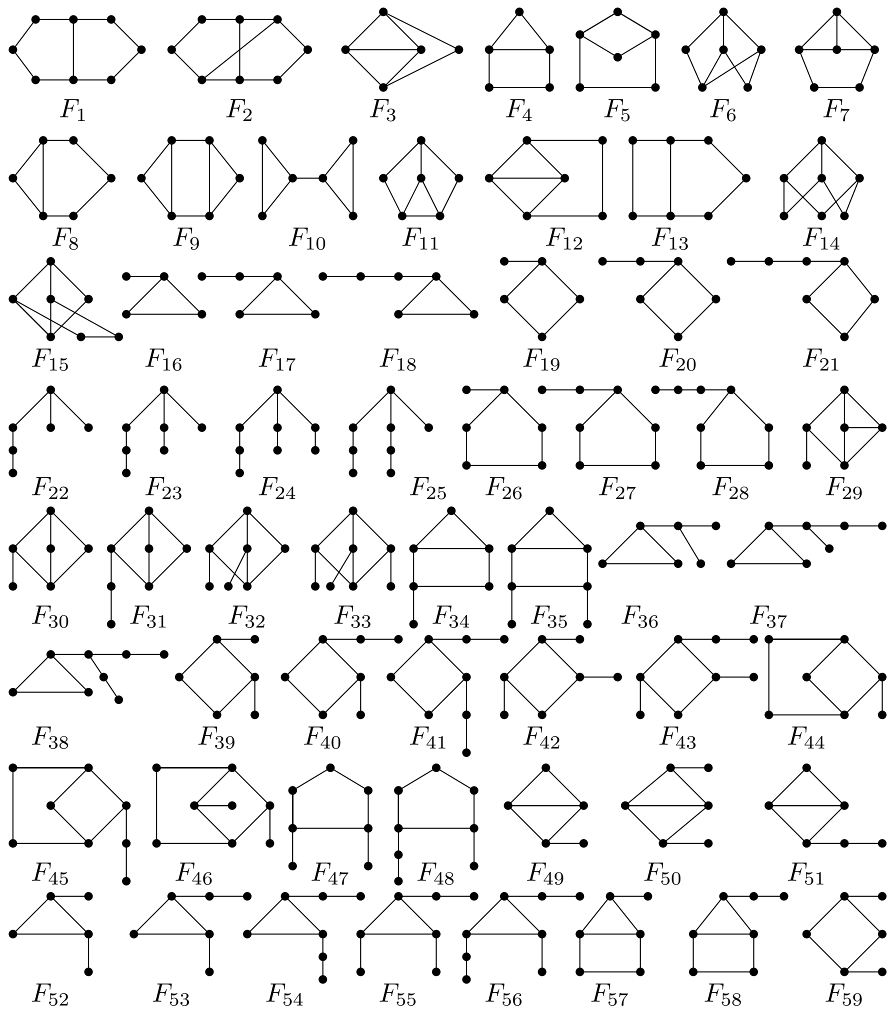

Figure 1: All graphs in the family $\mathcal{F}$.

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The graph $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_h, while a *wounded spider* S_w is the graph formed by subdividing at most $t-1$ of the edges of a star $K_{1,t}$ for $t \geq 2$. Clearly stars are wounded spiders. A *spider* is a healthy or wounded spider. The family $\mathcal{S}$ of the spider graphs obtained from $K_{1,3}$ is given in Figure 2.

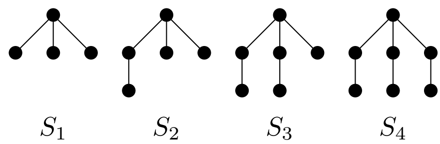

Figure 2: All graphs in the family $\mathcal{S}$.

Let $\mathcal{D}$ be the class of all graphs G that can be obtained from the double star $S(2,2)$ by subdividing each leaf of $S(2,2)$ at most once (see Figure 3).

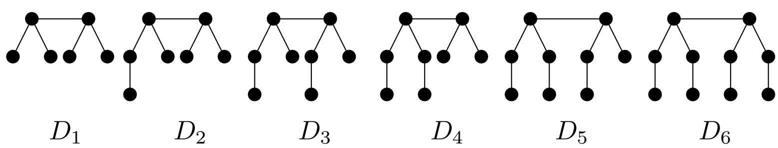

Figure 3: All graphs in the family $\mathcal{D}$.

Let $\mathcal{R} = \mathcal{F} \cup \mathcal{S} \cup \mathcal{D}$.

Theorem 2.1. *Let G be a graph of order n. Then $\gamma_R(G) = n-2$ if and only if $G \in \mathcal{R} \cup \{P_6, P_7, P_8, C_6, C_7, C_8, K_4, K_4 - e, K_{2,3}, K_{3,3}, P_2 \square P_3, P_2 \square C_3, cor(C_3), cor(C_4)\}$, where $e \in E(K_4)$.*

Proof. Let G be a graph of order n for which $\gamma_R(G) = n-2$. Proposition 1.1, implies that $\Delta(G) \leq 3$ and Proposition 1.2 implies that $n \leq 10$. Clearly $\Delta(G) > 1$. If $\Delta(G) = 2$ then G is a path or a cycle, and by Proposition 1.3 we can easily see that $G \in \{P_6, P_7, P_8, C_6, C_7, C_8\}$. Henceforth, we may assume $\Delta(G) = 3$. Let x be a vertex of maximum degree, and $N(x) = \{x_1, x_2, x_3\}$. We distinguish the following cases:

Case 1: $\delta(G) = 3$.

We deduce from Proposition 1.5 that $n \leq 6$. Since G is 3-regular, G has even order and so $n = 4$ or 6. If $n = 4$, then clearly $G \cong K_4$. Let $n = 6$

and $V(G) - N[x] = \{y, z\}$. If $yz \notin E(G)$, then obviously $G \cong K_{3,3}$ and if $yz \in E(G)$, then it is not hard to see that $G \cong P_2 \Box C_3$.

Case 2: $\delta(G) = 2$. By Proposition 1.4, we have $G \neq C_4, C_5$. If $G \cong F_1$ or $G \cong F_2$, then we are done. Let $G \notin \{F_1, F_2\}$. We conclude from Proposition 1.4 that $n \leq 7$. If $n = 4$, then clearly $G \cong K_4 - e$, where $e \in E(K_4)$, and if $n = 5$, then it is easy to verify that $G \in \{K_{2,3}, F_3, F_4\}$. If $n = 6$, then it follows from Proposition 2.2 and the fact $\Delta(G) = 3$ that $G \in \{P_2 \Box P_3, F_5, F_6, \ldots, F_{12}\}$. Assume that $n = 7$ and $V(G) - N[x] = \{u, v, w\}$. Then we have $\gamma_R(G) = 5$. Since $\deg(x) = 3$, we conclude from Proposition 2.2 (iv) that $\Delta(G[\{u, v, w\}]) \leq 1$. Since $N[y] \cup N[z] \subsetneqq V(G)$ for all pairs of vertices $y, z \in V(G)$, we deduce that $G \in \{F_{13}, F_{14}, F_{15}\}$.

Case 3: $\delta(G) = 1$. If there is a vertex $w \in V(G) - (N(x_1) \cup N(x_2) \cup N(x_3))$ with $\deg(w) \geq 2$, and $w_1, w_2 \in N(w)$, then $f = (\{x_1, x_2, x_3, w_1, w_2\}, V(G) - \{x, x_1, x_2, x_3, w, w_1, w_2\}, \{x, w\})$ is an RDF on G, implying that $\gamma_R(G) \leq n-3$, a contradiction. Thus every vertex of $V(G) - (N(x_1) \cup N(x_2) \cup N(x_3))$ is a leaf.

Consider x is the unique vertex of degree $\Delta(G)$. If $\deg(x_i) = 1$ for $i = 1, 2, 3$, then $G \cong S_1$. Thus suppose $\deg(x_1) = 2$. If x_1 is adjacent to x_2 or x_3 then $G \in \{F_{16}, F_{17}, F_{18}\}$. Thus suppose $x_1 \notin N(x_2) \cup N(x_3)$. In particular, $N(x)$ is an independent set. Let $y_1 \in N(x_1) - \{x\}$. Suppose $\deg(y_1) = 2$ and $y_1 x_2 \in E(G)$, then $G \in \{F_{19}, F_{20}, F_{21}\}$. Now suppose y_1 is a stem vertex adjacent to a leaf y_2. If $\deg(x_2) = \deg(x_3) = 1$ then $G \cong F_{22}$. Thus suppose $\deg(x_j) = 2$ for some $j = 2, 3$. Since any new leaf is within distance two from x when $deg(x_j) = 2$ for all $j = 2, 3$, and also any new leaf is within distance at most three from x when $deg(x_2) = 2$ and $deg(x_3) = 1$ (or $deg(x_2) = 1$ and $deg(x_3) = 2$), we obtain that $G \in \{F_{23}, F_{24}, F_{25}\}$.

Now suppose $y_2 x_2 \in E(G)$, then $G \in \{F_{26}, F_{27}, F_{28}\}$. Next, we suppose that $\deg(y_1) = 1$. If $\deg(x_2) = \deg(x_3) = 1$ then $G \cong S_2$. Thus suppose $\deg(x_2) = 2$. By the above argument we can suppose x_2 is a stem vertex. Now it is obvious that $G \cong S_3$. Finally suppose x_3 is a stem vertex. Now it is obvious that $G \cong S_4$.

Thus suppose G has more than one vertex of degree $\Delta(G)$. Let $y \neq x$ be a vertex of degree $\Delta(G)$. If $y \notin N(x)$, $N(x) \cap N(y) = \emptyset$ and $N(y) = \{y_1, y_2, y_3\}$, then $f = (\{x_1, x_2, x_3, y_1, y_2, y_3\}, V(G) - \{x, x_1, x_2, x_3, y, y_1, y_2, y_3\}, \{x, y\})$, is an RDF on G, implying that $\gamma_R(G) \leq n - 3$, a contradiction.

If $y \notin N(x)$ and $|N(x) \cap N(y)| = 1$, we let $x_1 \in N(x) \cap N(y)$. Let $N(y) = \{x_1, y_1, y_2\}$. Then $f = (\{x_1, x_2, x_3, y_1, y_2\}, V(G) - \{x, x_1, x_2, x_3, y, y_1, y_2\}, \{x, y\})$ is an RDF on G, implying that $\gamma_R(G) \leq n - 3$, a contradiction. Thus $|N(x) \cap N(y)| \geq 2$. We thus have the following claim.

Claim 1: $|N(a) \cap N(b)| \geq 2$ for every pair of non-adjacent vertices a and b of degree $\Delta(G)$.

Consider $|N(x) \cap N(y)| = 3$. Thus $N(x) = N(y) = \{x_1, x_2, x_3\}$. If there is a vertex $w \in V(G) - (N[x] \cup N[y])$ such that $deg(w) = 3$, then by Claim 1 and the assumption $\delta(G) = 1$, $|N(w) \cap \{x_1, x_2, x_3\}| = 2$. Consider $N(w) \cap \{x_1, x_2, x_3\} = \{x_1, x_2\}$ and $w_1 \in N(w) - \{x_1, x_2\}$. Then $f = (N(w) \cup \{x, y\}, V(G) - (N(w) \cup \{x, x_3, y, w\}), \{x_3, w\})$ is an RDF on G, implying that $\gamma_R(G) \leq n - 3$, a contradiction. Thus any vertex $w \in V(G) - N[x] \cup N[y]$ has degree at most two. The assumption $\Delta(G) = 3$ implies that $\Delta(G[\{x_1, x_2, x_3\}]) \leq 1$. Suppose $\Delta(G[\{x_1, x_2, x_3\}]) = 1$. We can suppose $x_1x_2 \in E(G)$. Since $\delta(G) = 1$, we have $deg(x_3) = 3$. Let $z_1 \in N(x_3) - \{x, y\}$. If z_1 is not leaf, and $z_2 \in N(z_1) - \{x_3\}$, then $f = (\{x, x_2, x_3, z_2, y\}, V(G) - \{x, x_1, x_2, x_3, z_1, z_2, y\}, \{x_1, z_1\})$ is an RDF on G, implying that $\gamma_R(G) \leq n - 3$, a contradiction. Thus z_1 is a leaf. Consequently $G \cong F_{29}$. Next suppose $\Delta(G[\{x_1, x_2, x_3\}]) = 0$. Since $\delta(G) = 1$, we can suppose $deg(x_1) = 3$. If $deg(x_2) = deg(x_3) = 2$ then $G \in \{F_{30}, F_{31}\}$. Thus suppose $deg(x_2) = 3$. Let $z_1 \in N(x_1) - \{x, y\}$ and $z_2 \in N(x_2) - \{x, y\}$.

Consider $deg(x_3) = 2$. If $z_1 = z_2$ then the assumption $\delta(G) = 1$ implies that z_1 is a stem vertex. Then assigning 2 to z_1 and x_3, and 0 to every other vertex gives an RDF of weight less than $n - 3$, a contradiction. Thus $z_1 \neq z_2$. Obviously, $N(x_1) \cap N(x_2) = \{x, y\}$. We show that z_1 and z_2 are leaves. Assume, to the contrary, that at least one of z_1 or z_2 is not a leaf, say z_1. Let $w \in N(z_1) - \{x_1\}$. Then $f = (\{x, x_1, y, y_1, w\}, V(G) - \{x, x_1, x_3, y, y_1, z_1, w\}, \{x_3, z_1\})$ is RDF on G, implying that $\gamma_R(G) \leq n - 3$, a contradiction. Thus z_1 and z_2 are leaves. Consequently $G \cong F_{32}$.

We thus assume for the next that $deg(x_3) = 3$. Let $z_3 \in N(x_3) - \{x, y\}$. By the assumption $\delta(G) = 1$ at most two of z_1, z_2 and z_3 are equal. Consider $z_1 = z_2$. Then assigning 2 to z_1 and x_3, 0 to every vertex in $N(z_1) \cup N(x_3)$, and 0 to every other vertex gives an RDF of weight less than $n - 3$, a contradiction. Thus no pair of z_1, z_2 and z_3 are equal. If $\deg(z_1) > 1$, then assigning 2 to z_1 and x_3, 0 to every vertex in $N(z_1) \cup N(x_3)$, and 0 to every other vertex gives an RDF of weight less than $n - 3$, a contradiction. Thus z_1 is a leaf, and similarly z_2 and z_3 are leaves. Consequently, $G \cong F_{33}$.

Thus for the next suppose $|N(x)\cap N(y)| = 2$. Furthermore, since any vertex of maximum degree can plays the role of x, we thus obtain the following.

Claim 2: $|N(a) \cap N(b)| = 2$ for every pair of non-adjacent vertices a and b of degree $\Delta(G)$. Suppose $xy \in E(G)$, (say $y = x_3$). Suppose now $N(y) \cap \{x_1, x_2\} = \emptyset$. Let $y_1, y_2 \in N(y) - \{x\}$. Suppose that $x_1 \in N(x_2)$.

Suppose that $\deg(x_2) = 3$. By Claim 2 we can suppose $y_1x_2 \in E(G)$. If $\deg(x_1) = 3$ then by Claim 2, $x_1 \in N(y_2)$. Now the assumption $\delta(G) = 1$ implies that $\deg(y_1) = 3$ or $\deg(y_2) = 3$, contradicting Claim 2. Thus $\deg(x_1) = 2$. Consequently, $G \in \{F_{34}, F_{35}\}$. Thus $\deg(x_2) = 2$, and similarly $\deg(x_1) = 2$. If $y_1 \in N(y_2)$ then by the assumption $\delta(G) = 1$ we can suppose $\deg(y_1) = 3$, contradicting Claim 2. Thus $y_1 \notin N(y_2)$. Consequently $G \in \{F_{36}, F_{37}, F_{38}\}$. Thus for the next suppose $x_1 \notin N(x_2)$. Consider $y_1 \in N(x_2)$. If $x_1 \in N(y_2)$ then by the assumption $\delta(G) = 1$ we can suppose $\deg(y_1) = 3$, contradicting Claim 2. Thus $x_1 \notin N(y_2)$.

Now we observe that $G \in \{cor(C_4), F_{39}, F_{40}, \ldots, F_{46}\}$. Thus suppose the set $\{x_1, x_2, y_1, y_2\}$ is independent, which implies that $G \in \{F_{47}, F_{48}\} \cup \mathcal{D}$. Thus suppose $N(y) \cap \{x_1, x_2\} \neq \emptyset$. Let $yx_1 \in E(G)$. If $yx_2 \in E(G)$, the assumption $\delta(G) = 1$ implies that some vertex in $N(x) \cap N(y)$ is of degree three. Consider $deg(x_1) = 3$. Let $z_1 \in N(x_1) - \{x, y\}$. Consider $deg(x_2) = 3$, and $z_2 \in N(x_2) - \{x, y\}$.

If $z_1 = z_2$ then z_1 is a stem vertex, and $G \cong F_{29}$. Thus $z_1 \neq z_2$. It can be seen that z_1 and z_2 are leaves. Consequently $G \cong F_{50}$. Next let $deg(x_2) = 2$. Now it can be seen that $G \in \{F_{49}, F_{51}\}$.

Thus y is not adjacent to x_2, and we can also suppose $x_1x_2 \notin E(G)$. It is not so difficult to check that, $\deg(x_2) \leq 2$. Let $y_1 \in N(y) - \{x, x_1\}$. If $\deg(y_1) = 3$, then by Claim 2, y_1 is adjacent to x_1, and $G \cong F_{50}$. Thus $\deg(y_1) \leq 2$, and similarly we can suppose each vertex in $N(x_1) - \{x, y\}$ has degree at most two.

Now it is straightforward to see that $G \in \{cor(C_3), F_{52}, F_{53}, \ldots, F_{58}\}$. Thus suppose $xy \notin E(G)$. We thus deduce that $\deg(x_i) \leq 2$ for $i \in \{1, 2, 3\}$. By Claim 2, $|N(x) \cap N(y)| = 2$. We can suppose $N(x) \cap N(y) = \{x_1, x_2\}$, and let $y_1 \in N(y) - \{x_1, x_2\}$. Clearly, y_1 is a leaf. Now $G \cong F_{59}$. □

3. Conclusion

In this paper, we have continued the study of Roman domination number. We have characterized all connected graphs G of order n whose Roman domination number is $n-1$ or $n-2$.

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